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2010 J. Phys. A: Math. Theor. 43 245201

(http://iopscience.iop.org/1751-8121/43/24/245201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 245201 (16pp) doi:10.1088/1751-8113/43/24/245201

Symmetries of systems of stochastic differential equations with diffusion matrices of full rank

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Received 3 February 2010, in final form 9 April 2010
Published 19 May 2010
Online at stacks.iop.org/JPhysA/43/245201

Abstract

Lie point symmetries of a system of stochastic differential equations (SDEs) with diffusion matrices of full rank are considered. It is proved that the maximal dimension of a symmetry group admitted by a system of n SDEs is $n + 2$. In addition, such systems cannot admit symmetry operators whose coefficients are proportional to a nonconstant coefficient of proportionality. These results are applied to compute the Lie group classification of a system of two SDEs. The classification is obtained with the help of non-equivalent realizations of real Lie algebras by fiber-preserving vector fields in $1 + 2$ variables. Possibilities of using symmetries for integration of SDEs by quadratures are discussed.

PACS numbers: 02.20.Sv, 02.30.Hq, 02.50.Ey

1. Introduction

The Lie group theory of differential equations is well understood [1–3]. It studies transformations by taking solutions of differential equations into other solutions of the same equations. Now this theory is a very general and useful tool for finding analytical solutions of large classes of differential equations.

Recently there appeared applications of Lie group theory to stochastic differential equations (SDEs). First, restricted cases of point transformations were considered [4–7]. Then, the theory for general point transformations was developed [8–12]. In the latter case the transformation of the Brownian motion needs to be more thoroughly specified.

In this paper we consider Lie point symmetries of systems of SDEs with diffusion matrices of full rank. For systems of n SDEs it is proved that the admitted symmetry group can be at most $n+2$ dimensional. The maximal dimension of the admitted symmetry group is achieved, for example, by systems with constant drift and diffusion coefficients. It is also shown that such systems cannot admit symmetry operators whose coefficients are proportional to a nonconstant coefficient of proportionality. These results can be used to carry out Lie group classifications.

The Lie group classification of a scalar stochastic ordinary differential equation (ODE) with one-dimensional Brownian motion was presented in [13]. The admitted symmetry group can be zero, one, two or three dimensional. To obtain this group classification a direct method was used. First, the SDE was simplified under an assumption that there exists one symmetry admitted by the equation. Then, all particular cases leading to the existence of additional symmetries were identified. This approach turns out to be too difficult for systems of SDEs.

Another possibility is to start from Lie algebras, which are known for low-dimensional cases, find their realizations by vector fields and obtain SDEs which admit these vector fields as symmetries. This method is used in the present paper to find the Lie point group classification of a system of two SDEs. In this case the symmetries are given by fiber-preserving vector fields in 1 + 2 dimensions.

As an application we consider the use of symmetries for integration of SDEs by quadratures. We obtain the classical theorems for symmetries acting in the space of dependent variables.

Symmetries of SDEs can be useful to find symmetries of the Fokker–Planck (FP) equation [7, 9]. In the case of fiber-preserving symmetries ($\tau = \tau(t)$), a symmetry of SDEs can be extended to a symmetry of the associated FP equation. The converse result holds only for operators satisfying an additional condition. In one spatial dimension, symmetries of a generic FP equation are known. The complete group classification of the linear (1 + 1)-dimensional homogeneous second-order parabolic equation was performed by Lie [14]. A modern treatment can be found in [1] (see also [15]). A number of papers are devoted specifically to the symmetries of the FP equation in one spatial dimension [16–18]. There are no general studies for higher dimensions. The existing results are limited to a special case of Kramers' equation for the diffusion matrix which is constant and degenerate [19], and the FP equation with a constant and positive definite diffusion matrix [20]. Both papers are restricted to FP equations in two spatial dimensions.

It should be noted that this paper deals with infinitesimal Lie group transformations which preserve the form of SDEs. The reconstruction of finite transformations from infinitesimal ones was discussed in [10, 12]. Generally, it is not guaranteed that the finite transformations, which are recovered from infinitesimal transformations, transform solutions of SDEs into another solutions.

This paper is organized as follows. In section 2 background information concerning symmetries of SDEs is provided. We also characterize symmetry properties for systems of SDEs which have diffusion matrices of full rank. Section 3 is devoted to the group classification of a system of two SDEs. We comment on the use of symmetries for integrability of SDEs by quadratures in section 4.

2. Systems of SDEs and symmetries

Let us consider a system of Itô SDEs,

$$\mathrm{d}x_i = f_i(t,\mathbf{x})\,\mathrm{d}t + g_{i\alpha}(t,\mathbf{x})\,\mathrm{d}W_\alpha(t), \qquad i = 1,\ldots,n, \quad \alpha = 1,\ldots,m, \tag{2.1}$$

where $f_i(t,\mathbf{x})$ is a drift vector, $g_{i\alpha}(t,\mathbf{x})$ is a diffusion matrix and $W_\alpha(t)$ is a vector Wiener process [21, 22]. Here and below we assume summation over repeated indexes. We restrict ourselves to the case

$$\mathrm{rank}\{g_{i\alpha}(t,\mathbf{x})\} = n, \tag{2.2}$$

i.e. the case when the diffusion matrix has full rank. In particular, this implies $m \geqslant n$.

2.1. Determining equations

We will be interested in infinitesimal group transformations (near identity changes of variables)

$$\bar{t} = \bar{t}(t, \mathbf{x}, a) \approx t + \tau(t, \mathbf{x})a, \qquad \bar{x}_i = \bar{x}_i(t, \mathbf{x}, a) \approx x_i + \xi_i(t, \mathbf{x})a, \tag{2.3}$$

which leave equations (2.1) and framework of Itô calculus invariant. Such transformations can be represented by generating operators of the form

$$X = \tau(t, \mathbf{x})\frac{\partial}{\partial t} + \xi_i(t, \mathbf{x})\frac{\partial}{\partial x_i}. \tag{2.4}$$

The determining equations for admitted symmetries [9] are

$$\frac{\partial \xi_i}{\partial t} + f_j\frac{\partial \xi_i}{\partial x_j} - \xi_j\frac{\partial f_i}{\partial x_j} - \tau\frac{\partial f_i}{\partial t} - f_i\frac{\partial \tau}{\partial t} - f_i f_j\frac{\partial \tau}{\partial x_j} - \frac{1}{2}f_i g_{j\alpha}g_{k\alpha}\frac{\partial^2 \tau}{\partial x_j \partial x_k} + \frac{1}{2}g_{j\alpha}g_{k\alpha}\frac{\partial^2 \xi_i}{\partial x_j \partial x_k} = 0, \tag{2.5}$$

$$g_{j\alpha}\frac{\partial \xi_i}{\partial x_j} - \xi_k\frac{\partial g_{i\alpha}}{\partial x_k} - \tau\frac{\partial g_{i\alpha}}{\partial t} - \frac{g_{i\alpha}}{2}\left(\frac{\partial \tau}{\partial t} + f_j\frac{\partial \tau}{\partial x_j} + \frac{1}{2}g_{j\beta}g_{k\beta}\frac{\partial^2 \tau}{\partial x_j \partial x_k}\right) = 0, \tag{2.6}$$

$$g_{j\alpha}\frac{\partial \tau}{\partial x_j} = 0. \tag{2.7}$$

It is interesting to note that the determining equations are deterministic even though they describe symmetries of a system of SDEs.

In the general case, when the functions $f_i(t, \mathbf{x})$ and $g_{i\alpha}(t, \mathbf{x})$ are arbitrary, the determining equations (2.5)–(2.7) have no non-trivial solutions, i.e. there are no symmetries.

Under condition (2.2) the last set of determining equations (2.7) can be solved as

$$\tau = \tau(t). \tag{2.8}$$

Therefore, the symmetries admitted by the system (2.1), (2.2) are fiber-preserving symmetries

$$X = \tau(t)\frac{\partial}{\partial t} + \xi_i(t, \mathbf{x})\frac{\partial}{\partial x_i} \tag{2.9}$$

that substantially simplify further consideration. In particular, we are restricted to equivalence transformations

$$\bar{t} = \bar{t}(t), \qquad \bar{\mathbf{x}} = \bar{\mathbf{x}}(t, \mathbf{x}), \qquad \bar{t}_t \neq 0, \qquad \det\left(\frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{x}}\right) \neq 0, \tag{2.10}$$

where the change of time is not random. According to the general result concerning the random time change in Brownian motion [22], the Brownian motion is transformed as

$$\mathrm{d}\bar{W}(\bar{t}) = \sqrt{\frac{\mathrm{d}\bar{t}(t)}{\mathrm{d}t}}\,\mathrm{d}W(t). \tag{2.11}$$

It should be noted that transformations (2.10) do not change the rank of the diffusion matrix. In particular, if the original system of SDEs has a diffusion matrix of full rank, then the diffusion matrix of the transformed system also has full rank.

Remark 2.1. Because the symmetries admitted by the system (2.1), (2.2) are fiber-preserving symmetries (2.9), they form a Lie algebra. It was shown in [9] that symmetries of Stratonovich systems always form Lie algebras. In a particular case $\tau = \tau(t)$ the determining equations for corresponding Itô and Stratonovich systems are identical. Therefore, all results of this paper established for Itô SDEs (2.1) are also valid for the corresponding Stratonovich SDEs:

$$\mathrm{d}x_i = h_i(t, \mathbf{x})\,\mathrm{d}t + g_{i\alpha}(t, \mathbf{x}) \circ \mathrm{d}W_\alpha(t), \qquad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m \tag{2.12}$$

with

$$h_i = f_i - \frac{1}{2} g_{k\alpha} \frac{\partial g_{i\alpha}}{\partial x_k}.$$

2.2. Symmetry properties

The considered systems have a bound on the dimension of the admitted symmetry group. We assume that all functions $f_i(t, \mathbf{x})$ and $g_{i\alpha}(t, \mathbf{x})$, describing the SDEs, as well as coefficients $\tau(t, \mathbf{x})$ and $\xi_i(t, \mathbf{x})$ of the symmetry operators are analytic.

Theorem 2.2. *The maximal dimension of a symmetry group admitted by the system of SDEs (2.1), (2.2) is $n + 2$.*

Proof. Let us write down a simplified version of the determining equations

$$\frac{\partial \xi_i}{\partial t} + f_j \frac{\partial \xi_i}{\partial x_j} - \xi_j \frac{\partial f_i}{\partial x_j} - \tau \frac{\partial f_i}{\partial t} - f_i \frac{\partial \tau}{\partial t} + \frac{1}{2} g_{j\alpha} g_{k\alpha} \frac{\partial^2 \xi_i}{\partial x_j \partial x_k} = 0, \tag{2.13}$$

$$g_{j\alpha} \left(\frac{\partial \xi_i}{\partial x_j} - \frac{\delta_{ij}}{2} \frac{\partial \tau}{\partial t} \right) = \xi_k \frac{\partial g_{i\alpha}}{\partial x_k} + \tau \frac{\partial g_{i\alpha}}{\partial t}, \tag{2.14}$$

$$\frac{\partial \tau}{\partial x_j} = 0, \tag{2.15}$$

where δ_{ij} is the Kronecker symbol. The set of equations (2.14) can be resolved as

$$\frac{\partial \xi_i}{\partial x_j} - \frac{\delta_{ij}}{2} \frac{\partial \tau}{\partial t} = \chi_{ij}, \qquad \chi_{ij} \in \operatorname{span}(\tau, \xi). \tag{2.16}$$

By $\operatorname{span}(\tau, \xi)$ we mean functions which are linear in τ and $\xi = \{\xi_k\}_{k=0}^n$ with coefficients depending on some functions of t and $\mathbf{x}$. If the system (2.14) is overdetermined, there will be additional constraints.

From (2.16) we obtain

$$\frac{\partial^2 \xi_i}{\partial x_j \partial x_k} = \varphi_{ijk}, \qquad \varphi_{ijk} \in \operatorname{span}\left(\tau, \frac{\partial \tau}{\partial t}, \xi\right). \tag{2.17}$$

Substitution of (2.16) and (2.17) into equations (2.13) provides us with

$$\frac{\partial \xi_i}{\partial t} = \varphi_{i0}, \qquad \varphi_{i0} \in \operatorname{span}\left(\tau, \frac{\partial \tau}{\partial t}, \xi\right). \tag{2.18}$$

Finally, from (2.15), (2.16) and (2.18) we conclude that all derivatives of τ and ξ are linear homogeneous functions of τ, ξ and τ_t. The total number of unconstrained derivatives is at most $n + 2$. Thus, the space of the solutions is at most $n + 2$ dimensional. A detailed justification of this reasoning can be found in section 48 of [23].

Let us show that systems

$$\mathrm{d}x_i = C_i \mathrm{d}t + C_{i\alpha}\,\mathrm{d}W_\alpha(t), \qquad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m \tag{2.19}$$

with constant drift and diffusion coefficients admit symmetry groups of maximal dimension $n+2$. We recall that equations (2.15) give us $\tau = \tau(t)$. In the case of a constant diffusion matrix equations (2.14) take the form

$$C_{j\alpha}\left(\frac{\partial \xi_i}{\partial x_j} - \frac{\delta_{ij}}{2}\tau'(t)\right) = 0,$$

and can be solved as

$$\xi_i = \tfrac{1}{2}\tau'(t)x_i + A_i(t),$$

where $A_i(t)$ are the arbitrary functions. Substitution into the set of equations (2.13) leads to

$$\tfrac{1}{2}\tau''(t)x_i + A_i'(t) - \tfrac{1}{2}C_i\tau'(t) = 0.$$

The solution is

$$\tau = \alpha t + \beta, \qquad \xi_i = \frac{\alpha}{2}(x_i + C_i t) + \gamma_i,$$

where α, β and γ_i are arbitrary constants. The symmetry group is given by the operators

$$X = \frac{\partial}{\partial t}, \qquad Y_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \qquad Z = 2t\frac{\partial}{\partial t} + (x_i + C_i t)\frac{\partial}{\partial x_i}. \tag{2.20}$$

Let us note that these operators have the algebra structure

$$[X, Y_i] = 0, \qquad [X, Z] = 2X + C_i Y_i, \qquad [Y_i, Z] = Y_i. \qquad \square$$

By the change of variables $\bar{x}_i = x_i - C_i t$ we can always remove the drift terms. We obtain the system

$$\mathrm{d}\bar{x}_i = C_{i\alpha}\,\mathrm{d}W_\alpha(t), \qquad i = 1, \dots, n, \quad \alpha = 1, \dots, m. \tag{2.21}$$

For $m = n$ this system can be split into n separate equations of Brownian motion

$$\mathrm{d}\bar{x}_i = \mathrm{d}W_i(t), \qquad i = 1, \dots, n,$$

by an appropriate linear transformation of dependent variables.

Similarly, we can establish a bound on the dimension of the admitted symmetry group acting in the space of dependent variables.

Theorem 2.3. *Let us consider group transformations generated by the operators of the form*

$$X = \xi_i(t, \mathbf{x})\frac{\partial}{\partial x_i}. \tag{2.22}$$

The maximal dimension of such a symmetry group admitted by the system of SDEs (2.1), (2.2) is n.

To facilitate Lie group classifications we will show that the system (2.1), (2.2) cannot admit symmetry operators whose coefficients are proportional to a nonconstant coefficient of proportionality. It can be done with the help of first integrals.

Definition 2.4. *A quantity $I(t, \mathbf{x})$ is a first integral of a system of SDEs (2.1) if it remains constant on the solutions of SDEs.*

The Itô differential of the first integral

$$\mathrm{d}I = \left(\frac{\partial I}{\partial t} + f_j\frac{\partial I}{\partial x_j} + \frac{1}{2}g_{j\alpha}g_{k\alpha}\frac{\partial^2 I}{\partial x_j \partial x_k}\right)\mathrm{d}t + g_{j\alpha}\frac{\partial I}{\partial x_j}\,\mathrm{d}W_\alpha(t) = 0 \tag{2.23}$$

leads to partial differential equations

$$D_0(I) = \frac{\partial I}{\partial t} + f_j \frac{\partial I}{\partial x_j} + \frac{1}{2} g_{j\alpha} g_{k\alpha} \frac{\partial^2 I}{\partial x_j \partial x_k} = 0, \tag{2.24}$$

$$D_\alpha(I) = g_{j\alpha} \frac{\partial I}{\partial x_j} = 0, \qquad \alpha = 1, \ldots, m, \tag{2.25}$$

which a conserved quantity should satisfy.

Proposition 2.5. *A system of SDEs (2.1) (2.2) has no first integrals.*

Proof. Under condition (2.2) the set of equations (2.25) has only time-dependent solutions $I(t)$. Substituting $I(t)$ into equation (2.24), we obtain $I \equiv \text{const}$. □

Relations of first integrals and symmetries of SDEs were investigated in a number of papers [4–6, 9]. We will rely on the following property.

Theorem 2.6. *A system of SDEs (2.1) with a non-zero diffusion matrix admits two linearly connected symmetries*

$$X_1 = \tau(t, \mathbf{x}) \frac{\partial}{\partial t} + \xi_i(t, \mathbf{x}) \frac{\partial}{\partial x_i}$$

and

$$X_2 = I(t, \mathbf{x}) \tau(t, \mathbf{x}) \frac{\partial}{\partial t} + I(t, \mathbf{x}) \xi_i(t, \mathbf{x}) \frac{\partial}{\partial x_i}$$

if and only if the function $I(t, \mathbf{x})$ is a first integral of the system.

Proof. Let us show that if the system (2.1) admits the symmetries X_1 and X_2, then $I(t, \mathbf{x})$ is a first integral. The determining equations (2.5)–(2.7) hold for both operators X_1 and X_2. From these sets of equations we obtain

$$(\xi_i - f_i \tau) D_0(I) + (D_\alpha(\xi_i) - f_i D_\alpha(\tau)) D_\alpha(I) = 0, \tag{2.26}$$

$$\xi_i D_\alpha(I) - \frac{g_{i\alpha}}{2} (\tau D_0(I) + D_\beta(\tau) D_\beta(I)) = 0, \tag{2.27}$$

$$\tau D_\alpha(I) = 0. \tag{2.28}$$

If $\tau \neq 0$, we obtain equations (2.25) from (2.28). Then, since there exists at least one element $g_{j\alpha} \neq 0$, we obtain equation (2.24) from (2.27). The last set of equations (2.26) is not required.

If $\tau = 0$, there exists at least one coefficient $\xi_i \neq 0$. Equations (2.28) hold identically. We obtain equations (2.25) from set (2.27) and equation (2.24) from set (2.26).

Inversely, it is easy to show that if the operator X_1 satisfies the determining equations and $I(t, \mathbf{x})$ is a first integral, then the operator X_2 also satisfies the determining equations, i.e. it is also a symmetry of the system of SDEs. □

Therefore, if the system (2.1) has at least one symmetry and one first integral, it admits an infinite-dimensional symmetry group.

Corollary 2.7. *A system of SDEs (2.1), (2.2) does not admit linearly connected symmetry operators (symmetry operators whose coefficients are proportional to a nonconstant coefficient of proportionality).*

Proof. By proposition 2.5 the system (2.1), (2.2) does not possess first integrals. It follows from theorem 2.6 that it cannot admit linearly connected symmetries. □

In the following sections we will construct a group classification using realizations of real Lie algebras by non-vanishing vector fields. Corollary 2.7 will be very useful to discard realizations which cannot be admitted as symmetries.

3. Group classification of a system of two SDEs

In this section we consider a particular case of the system (2.1), (2.2) corresponding to $n = 2$, namely the system

$$\begin{aligned} \mathrm{d}x_1 &= f_1(t, x_1, x_2)\,\mathrm{d}t + g_{1\alpha}(t, x_1, x_2)\,\mathrm{d}W_\alpha(t), \\ \mathrm{d}x_2 &= f_2(t, x_1, x_2)\,\mathrm{d}t + g_{2\alpha}(t, x_1, x_2)\,\mathrm{d}W_\alpha(t), \end{aligned} \qquad \alpha = 1, \ldots, m, \tag{3.1}$$

satisfying the condition

$$\operatorname{rank} \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \end{pmatrix} = 2. \tag{3.2}$$

Condition (3.2) requires $m \geqslant 2$.

From section 2 it is known that the system (3.1), (3.2) can admit only fiber-preserving symmetries

$$X = \tau(t)\frac{\partial}{\partial t} + \xi_1(t, x_1, x_2)\frac{\partial}{\partial x_1} + \xi_2(t, x_1, x_2)\frac{\partial}{\partial x_2}. \tag{3.3}$$

The maximal dimension of the admitted symmetry group is 4. Equivalence transformations (2.10) take the form

$$\begin{aligned} &\bar{t} = \bar{t}(t), \qquad \bar{x}_1 = \bar{x}_1(t, x_1, x_2), \qquad \bar{x}_2 = \bar{x}_2(t, x_1, x_2), \\ &\bar{t}_t \neq 0, \qquad \det \frac{\partial(\bar{x}_1, \bar{x}_2)}{\partial(x_1, x_2)} \neq 0. \end{aligned} \tag{3.4}$$

To obtain a group classification it is convenient to start from Lie algebras. Given their structure constants, we find non-vanishing vector fields satisfying the commutator relations. Thus, we find all possible realizations of the Lie algebras. Two realizations of the same Lie algebra are considered equivalent if there exists an equivalence transformation of the form (3.4) mapping one of the realizations into the other.

We will construct non-equivalent realizations of one-, two- and three-dimensional real Lie algebras by non-vanishing vector fields (3.3). Such realizations were considered in [24] to carry out the group classification of a nonlinear heat conductivity equation. Since the authors do not provide a list of all realizations of Lie algebras (they discard the realizations which cannot be admitted by the nonlinear PDE already in the two-dimensional case) we repeat the construction procedure. It is convenient to follow the description of real Lie algebras given in [25]. To make the paper self-sufficient we provide construction of one- and two-dimensional realizations and comment on the construction of three-dimensional realizations. The procedure can be continued for Lie algebras of dimension four. However, it turns out that for our purpose it is better to take systems invariant with respect to three-dimensional symmetry groups and investigate them for additional symmetries by direct computation.

3.1. One-dimensional symmetry groups

A one-dimensional algebra is represented by operator (3.3). By change of variables (3.4) it can be brought to the form

$$X_1 = \frac{\partial}{\partial t} \qquad \text{if} \quad \tau(t) \neq 0 \tag{3.5}$$

Table 1. Realizations of one- and two-dimensional real Lie algebras by vector fields (3.3) up to equivalence transformations (3.4).

Algebra	Rank of realization	N	Realization
A_1	1	1	$X_1 = \frac{\partial}{\partial t}$
		2	$X_1 = \frac{\partial}{\partial x_1}$
$2A_1$	2	1	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_1}$
$[X_1, X_2] = 0$		2	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}$
	1	3	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = t\frac{\partial}{\partial x_1}$
		4	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_2\frac{\partial}{\partial x_1}$
$A_{2.1}$	2	1	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1}$
$[X_1, X_2] = X_1$		2	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1}$
		3	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$
	1	4	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t}$
		5	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1\frac{\partial}{\partial x_1}$

or

$$X_1 = \frac{\partial}{\partial x_1} \qquad \text{if} \quad \tau(t) = 0. \tag{3.6}$$

We present these non-equivalent realizations in table 1 and the corresponding invariant system of two SDEs in table 2. It should be noted that table 2 contains the most general forms of the invariant systems of SDEs, which allow simplifying transformations.

3.2. Two-dimensional symmetry groups

Given a realization of a two-dimensional Lie algebra by two vector fields (3.3), we can transform one of these vector fields to the form (3.5) or (3.6).

Let us start with the case when X_1 is operator (3.5). The possible equivalence transformations are restricted to

$$\bar{t} = t + \alpha, \qquad \bar{x}_1 = \bar{x}_1(x_1, x_2), \qquad \bar{x}_2 = \bar{x}_2(x_1, x_2), \qquad \det \frac{\partial(\bar{x}_1, \bar{x}_2)}{\partial(x_1, x_2)} \neq 0, \tag{3.7}$$

where α is an arbitrary constant. There are two possibilities for the second operator X_2 of the two-dimensional Lie algebra.

(1) $[X_1, X_2] = 0$

In this case the most general form of the second operator is

$$X_2 = C_1\frac{\partial}{\partial t} + \xi_1(x_1, x_2)\frac{\partial}{\partial x_1} + \xi_2(x_1, x_2)\frac{\partial}{\partial x_2},$$

where $\xi_1(x_1, x_2)$ and $\xi_2(x_1, x_2)$ are the arbitrary functions. The constant C_1 can be removed by changing the second operator $X_2 \to X_2 - C_1X_1$. By the change of variables (3.7) this operator can be simplified as

$$X_2 = \frac{\partial}{\partial x_1}.$$

Table 2. Lie group classification of systems of two SDEs admitting one- and two-dimensional symmetry groups. f_i and $g_{i\alpha}$ are the arbitrary functions of their arguments such that there exist α and β for which $g_{1\alpha}g_{2\beta} - g_{1\beta}g_{2\alpha} \not\equiv 0$. The table contains the most general forms of the invariant systems of SDEs, which allow further simplification by the equivalence transformations.

Algebra	Realization	System of SDEs
A_1	$X_1 = \frac{\partial}{\partial t}$	$dx_1 = f_1(x_1, x_2)\,dt + g_{1\alpha}(x_1, x_2)\,dW_\alpha(t)$ $dx_2 = f_2(x_1, x_2)\,dt + g_{2\alpha}(x_1, x_2)\,dW_\alpha(t)$
	$X_1 = \frac{\partial}{\partial x_1}$	$dx_1 = f_1(t, x_2)\,dt + g_{1\alpha}(t, x_2)\,dW_\alpha(t)$ $dx_2 = f_2(t, x_2)\,dt + g_{2\alpha}(t, x_2)\,dW_\alpha(t)$
$2A_1$ $[X_1, X_2] = 0$	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_1}$	$dx_1 = f_1(x_2)\,dt + g_{1\alpha}(x_2)\,dW_\alpha(t)$ $dx_2 = f_2(x_2)\,dt + g_{2\alpha}(x_2)\,dW_\alpha(t)$
	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}$	$dx_1 = f_1(t)\,dt + g_{1\alpha}(t)\,dW_\alpha(t)$ $dx_2 = f_2(t)\,dt + g_{2\alpha}(t)\,dW_\alpha(t)$
$A_{2.1}$ $[X_1, X_2] = X_1$	$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1}$	$dx_1 = f_1(x_2)\,dt + g_{1\alpha}(x_2)\sqrt{x_1}\,dW_\alpha(t)$ $dx_2 = \frac{f_2(x_2)}{x_1}\,dt + \frac{g_{2\alpha}(x_2)}{\sqrt{x_1}}\,dW_\alpha(t)$
	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1}$	$dx_1 = f_1(x_2)\,dt + g_{1\alpha}(x_2)\sqrt{t}\,dW_\alpha(t)$ $dx_2 = \frac{f_2(x_2)}{t}\,dt + \frac{g_{2\alpha}(x_2)}{\sqrt{t}}\,dW_\alpha(t)$
	$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$	$dx_1 = f_1(t)x_2\,dt + g_{1\alpha}(t)x_2\,dW_\alpha(t)$ $dx_2 = f_2(t)x_2\,dt + g_{2\alpha}(t)x_2\,dW_\alpha(t)$

(2) $[X_1, X_2] = X_1$
In this case we obtain

$$X_2 = (t + C_1)\frac{\partial}{\partial t} + \xi_1(x_1, x_2)\frac{\partial}{\partial x_1} + \xi_2(x_1, x_2)\frac{\partial}{\partial x_2}.$$

An arbitrary constant C_1 can be discarded. Then, the operator can be brought to the form

$$X_2 = t\frac{\partial}{\partial t}, \qquad \text{if} \quad \xi_1(x_1, x_2) = 0 \quad \text{and} \quad \xi_1(x_1, x_2) = 0$$

or

$$X_2 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1}, \qquad \text{if} \quad \xi_1(x_1, x_2) \neq 0 \quad \text{or} \quad \xi_1(x_1, x_2) \neq 0.$$

We repeat this procedure for the other realization X_1, which is given by operator (3.6). It is preserved by equivalence transformations

$$\bar{t} = \bar{t}(t), \qquad \bar{x}_1 = x_1 + f(t, x_2), \qquad \bar{x}_2 = g(t, x_2), \qquad \bar{t}_t \neq 0, \quad g_{x_2} \neq 0. \tag{3.8}$$

(1) For $[X_1, X_2] = 0$ we obtain

$$X_2 = \tau(t)\frac{\partial}{\partial t} + \xi_1(t, x_2)\frac{\partial}{\partial x_1} + \xi_2(t, x_2)\frac{\partial}{\partial x_2},$$

which can be simplified to the form

$$X_2 = \frac{\partial}{\partial t} \qquad \text{if} \quad \tau(t) \neq 0,$$

$$X_2 = \frac{\partial}{\partial x_2} \qquad \text{if} \quad \tau(t) = 0 \quad \text{and} \quad \xi_2(t, x_2) \neq 0,$$

$$X_2 = x_2\frac{\partial}{\partial x_1} \quad \text{or} \quad X_2 = t\frac{\partial}{\partial x_1} \qquad \text{if} \quad \tau(t) = 0 \quad \text{and} \quad \xi_2(t, x_2) = 0.$$

(2) For $[X_1, X_2] = X_1$ the operator X_2 has the form

$$X_2 = \tau(t)\frac{\partial}{\partial t} + (x_1 + \xi_1(t, x_2))\frac{\partial}{\partial x_1} + \xi_2(t, x_2)\frac{\partial}{\partial x_2}.$$

It can transformed to a simpler form

$$X_2 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} \qquad \text{if} \quad \tau(t) \neq 0,$$

$$X_2 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} \qquad \text{if} \quad \tau(t) = 0 \quad \text{and} \quad \xi_2(t, x_2) \neq 0$$

or

$$X_2 = x_1\frac{\partial}{\partial x_1} \qquad \text{if} \quad \tau(t) = 0 \quad \text{and} \quad \xi_2(t, x_2) = 0.$$

The non-equivalent realizations obtained are summarized in table 1. It follows from corollary 2.7 that linearly connected operators cannot be admitted by the systems (3.1), (3.2). This excludes four out of nine realizations of two-dimensional Lie algebras. The invariant systems of two SDEs, corresponding to the other realizations, are given in table 2.

3.3. Three-dimensional symmetry groups

The three-dimensional Lie algebras can be split into solvable and unsolvable. The solvable algebras and algebra

$$sl(2,\mathbb{R}): \qquad [X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3, \qquad [X_1, X_3] = 2X_2 \tag{3.9}$$

contain two-dimensional subalgebras. Their realizations can be constructed with the help of realizations of two-dimensional algebras. This procedure is similar to that outlined in the previous point, where realizations of two-dimensional algebras were obtained with the help of those for one-dimensional algebras. It turns out that there are many realizations which cannot be symmetries of systems (3.1), (3.2). In table 3 we provide only realizations which can be admitted by the system of two SDEs and the corresponding invariant systems. The table presents the most general forms of SDEs, which allow further simplification by equivalence transformations. Note that the table contains only non-zero commutators.

Remark 3.1. Let us note that the realizations

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial t}$$

are equivalent as realizations of an Abelian two-dimensional algebra. However, they are not equivalent when we construct realizations of three-dimensional algebras built on the two-dimensional ones. See, for example, the first and second realizations of algebra $A_{2.1} \oplus A_1$ or $A_{3.4}^a$ in table 3.

Remark 3.2. Two realizations of three-dimensional algebras which provide invariant systems of SDEs are not present in table 3. These cases are a realization of the Abelian Lie algebra $3A_1$, given by the operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x_1}, \qquad X_3 = \frac{\partial}{\partial x_2}, \tag{3.10}$$

and a realization of the algebra

$$A_{3.3}: \quad [X_1, X_2] = 0, \qquad [X_1, X_3] = X_1, \qquad [X_2, X_3] = X_2,$$

Table 3. Lie group classification of systems of two SDEs admitting three-dimensional symmetry groups. C_i and $C_{i\alpha}$ are the constants such that there exist α and β for which $C_{1\alpha}C_{2\beta}-C_{1\beta}C_{2\alpha}\neq 0$. The table contains the most general forms of the invariant systems of SDEs, which allow further simplification by the equivalence transformations.

Algebra	Realization	System of SDEs
$A_{2.1}\oplus A_1$ $[X_1,X_2]=X_1$	$X_1=\frac{\partial}{\partial t},\quad X_2=t\frac{\partial}{\partial t}+x_2\frac{\partial}{\partial x_2},$ $X_3=\frac{\partial}{\partial x_1}$	$\mathrm{d}x_1=\frac{C_1}{x_2}\,\mathrm{d}t+\frac{C_{1\alpha}}{\sqrt{x_2}}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{d}t+C_{2\alpha}\sqrt{x_2}\,\mathrm{d}W_\alpha(t)$
	$X_1=\frac{\partial}{\partial x_1},\quad X_2=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2},$ $X_3=\frac{\partial}{\partial t}$	$\mathrm{d}x_1=C_1x_2\,\mathrm{d}t+C_{1\alpha}x_2\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2x_2\,\mathrm{d}t+C_{2\alpha}x_2\,\mathrm{d}W_\alpha(t)$
	$X_1=\frac{\partial}{\partial x_1},\quad X_2=t\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_1},$ $X_3=\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=C_1\mathrm{d}t+C_{1\alpha}\sqrt{t}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=\frac{C_2}{t}\,\mathrm{d}t+\frac{C_{2\alpha}}{\sqrt{t}}\,\mathrm{d}W_\alpha(t)$
$A_{3.1}$ $[X_2,X_3]=X_1$	$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial t},$ $X_3=t\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=(x_2+C_1)\,\mathrm{d}t+C_{1\alpha}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{d}t+C_{2\alpha}\,\mathrm{d}W_\alpha(t)$
	$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},$ $X_3=\frac{\partial}{\partial t}+x_2\frac{\partial}{\partial x_1}$	$\mathrm{d}x_1=(C_2t+C_1)\,\mathrm{d}t$ $+(C_{2\alpha}t+C_{1\alpha})\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{d}t+C_{2\alpha}\,\mathrm{d}W_\alpha(t)$
$A_{3.2}$ $[X_1,X_3]=X_1,$ $[X_2,X_3]=X_1+X_2$	$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial t},$ $X_3=t\frac{\partial}{\partial t}+(t+x_1)\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=(x_2+C_1)\,\mathrm{d}t+C_{1\alpha}\,\mathrm{e}^{x_2/2}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{e}^{-x_2}\,\mathrm{d}t+C_{2\alpha}\,\mathrm{e}^{-x_2/2}\,\mathrm{d}W_\alpha(t)$
	$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},$ $X_3=\frac{\partial}{\partial t}+(x_1+x_2)\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=(C_2t+C_1)\,\mathrm{e}^t\,\mathrm{d}t$ $+(C_{2\alpha}t+C_{1\alpha})\,\mathrm{e}^t\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{e}^t\,\mathrm{d}t+C_{2\alpha}\,\mathrm{e}^t\,\mathrm{d}W_\alpha(t)$
$A_{3.3}$ $[X_1,X_3]=X_1,$ $[X_2,X_3]=X_2$	$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x_1},$ $X_3=t\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=C_1\mathrm{d}t+C_{1\alpha}\,\mathrm{e}^{x_2/2}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{e}^{-x_2}\,\mathrm{d}t+C_{2\alpha}\,\mathrm{e}^{-x_2/2}\,\mathrm{d}W_\alpha(t)$
$A^a_{3.4},$ $\|a\|\leqslant 1,\quad a\neq 0,1$ $[X_1,X_3]=X_1,$ $[X_2,X_3]=aX_2$	$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x_1},$ $X_3=t\frac{\partial}{\partial t}+ax_1\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=C_1\mathrm{e}^{(a-1)x_2}\,\mathrm{d}t$ $+C_{1\alpha}\,\mathrm{e}^{(a-1/2)x_2}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{e}^{-x_2}\,\mathrm{d}t+C_{2\alpha}\,\mathrm{e}^{-x_2/2}\,\mathrm{d}W_\alpha(t)$
	$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial t},$ $X_3=at\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=C_1\,\mathrm{e}^{(1-a)x_2}\,\mathrm{d}t$ $+C_{1\alpha}\,\mathrm{e}^{(1-a/2)x_2}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{e}^{-ax_2}\,\mathrm{d}t+C_{2\alpha}\,\mathrm{e}^{-ax_2/2}\,\mathrm{d}W_\alpha(t)$
	$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},$ $X_3=\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_1}+ax_2\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=C_1\,\mathrm{e}^t\,\mathrm{d}t+C_{1\alpha}\,\mathrm{e}^t\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=C_2\,\mathrm{e}^{at}\,\mathrm{d}t+C_{2\alpha}\,\mathrm{e}^{at}\,\mathrm{d}W_\alpha(t)$
$A^b_{3.5}, b\geqslant 0$ $[X_1,X_3]=bX_1-X_2,$ $[X_2,X_3]=X_1+bX_2$	$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},$ $X_3=\frac{\partial}{\partial t}+(bx_1+x_2)\frac{\partial}{\partial x_1}$ $+(-x_1+bx_2)\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=(C_1\sin t+C_2\cos t)\,\mathrm{e}^{bt}\,\mathrm{d}t$ $+(C_{1\alpha}\sin t+C_{2\alpha}\cos t)\,\mathrm{e}^{bt}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=(C_1\cos t-C_2\sin t)\,\mathrm{e}^{bt}\,\mathrm{d}t$ $+(C_{1\alpha}\cos t-C_{2\alpha}\sin t)\,\mathrm{e}^{bt}\,\mathrm{d}W_\alpha(t)$
$sl(2,\mathbb{R})$ $[X_1,X_2]=X_1,$ $[X_2,X_3]=X_3,$ $[X_1,X_3]=2X_2$	$X_1=\frac{\partial}{\partial t},\quad X_2=t\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_1},$ $X_3=t^2\frac{\partial}{\partial t}+2tx_1\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2}$	$\mathrm{d}x_1=(2x_2+C_1)\,\mathrm{d}t+C_{1\alpha}\sqrt{x_1}\,\mathrm{d}W_\alpha(t)$ $\mathrm{d}x_2=\frac{x_2^2+C_1x_2+C_2}{x_1}\,\mathrm{d}t+\frac{C_{1\alpha}x_2+C_{2\alpha}}{\sqrt{x_1}}\,\mathrm{d}W_\alpha(t)$

given by the operators

$$X_1=\frac{\partial}{\partial x_1},\qquad X_2=\frac{\partial}{\partial x_2},\qquad X_3=2t\frac{\partial}{\partial t}+x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}. \tag{3.11}$$

In both cases we obtain systems of SDEs which actually admit four symmetries. These systems are equivalent to the system (3.13). Symmetry operators (3.10) and (3.11) are subgroups of operators (3.14).

The other unsolvable three-dimensional algebra

$$so(3): \quad [X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2 \tag{3.12}$$

requires a special consideration.

Proposition 3.3 [24]. *Within the equivalence class (3.4), there exists only one realization of the algebra $so(3)$ by operators of the form (3.3), which can be chosen as*

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = -\sin x_1 \tan x_2 \frac{\partial}{\partial x_1} - \cos x_1 \frac{\partial}{\partial x_2},$$
$$X_3 = -\cos x_1 \tan x_2 \frac{\partial}{\partial x_1} + \sin x_1 \frac{\partial}{\partial x_2}.$$

Direct verification shows that there is no invariant system of two SDEs for this set of operators.

3.4. Four-dimensional symmetry groups

According to theorem 2.2 a symmetry group admitted by the system (3.1), (3.2) is at most four dimensional. It is possible to avoid construction of realizations of all four-dimensional Lie algebras if we take into account that any four-dimensional algebra contains a three-dimensional subalgebra. We can take systems of SDEs admitting three symmetries and investigate them for an additional symmetry by direct computation. There is only one (up to equivalence) invariant system of two SDEs which admits four symmetries. It is the system

$$\mathrm{d}x_1 = C_{1\alpha}\,\mathrm{d}W_\alpha(t), \qquad \mathrm{d}x_2 = C_{2\alpha}\,\mathrm{d}W_\alpha(t), \tag{3.13}$$

which admits symmetries

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x_1}, \qquad X_3 = \frac{\partial}{\partial x_2}, \qquad X_4 = 2t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}. \tag{3.14}$$

These operators correspond to a particular case of algebra $A_{4.5}$ [25], which has non-zero commutators

$$[X_1, X_4] = 2X_1, \qquad [X_2, X_4] = X_2, \qquad [X_3, X_4] = X_3.$$

For $m = 2$ this system can be split into two separate equation of Brownian motion

$$\mathrm{d}x_1 = \mathrm{d}W_1(t), \qquad \mathrm{d}x_2 = \mathrm{d}W_2(t)$$

by an appropriate linear transformation of the dependent variables.

The results of this section can be summed up as a theorem.

Theorem 3.4. *Let a system of SDEs (3.1), (3.2) be invariant under the Lie group G of local point transformations with the Lie algebra realized by the vector fields $X_1, \ldots, X_k$ of the form (3.3). Then, $k = 0, 1, 2, 3$ or 4 and*

$$\mathrm{rank}(X_1, \ldots, X_k) = \min(k, 3).$$

Probably, the theorem can be extended to a more general case of systems of SDEs (2.1), (2.2).

4. Integrability by quadratures

It is known (see, for example, [3]) that knowledge of an r-parameter solvable group of symmetries allows us to reduce the order of a system of first-order ODEs by r. An analogous result is valid for the system of SDEs. However, in the general case we can use only symmetries acting in the space of the dependent variables. This can be illustrated for a scalar SDE.

Example 4.1. Let us consider a SDE

$$\mathrm{d}x = f(t,x)\,\mathrm{d}t + g(t,x)\,\mathrm{d}W(t), \qquad g(t,x) \not\equiv 0. \tag{4.1}$$

In [13] it was shown that if equation (4.1) admits a symmetry

$$X = \xi(t,x)\frac{\partial}{\partial x},$$

it can be transformed into the form

$$\mathrm{d}x = f(t)\,\mathrm{d}t + g(t)\,\mathrm{d}W(t),$$

which can be solved as

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s)\,\mathrm{d}s + \int_{t_0}^{t} g(s)\,\mathrm{d}W(s).$$

However, if the admitted symmetry is

$$X = \tau(t)\frac{\partial}{\partial t} + \xi(t,x)\frac{\partial}{\partial x}, \qquad \tau(t) \not\equiv 0,$$

then the equation can be brought to the form

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}W(t),$$

which is not integrable in the general case.

The classical results concerning the first-order system of ODEs take the following form. It is also valid for the system which does not satisfy rank restriction (2.2).

Theorem 4.1. *Suppose the system (2.1) admits a symmetry of the form*

$$X = \xi_i(t,x)\frac{\partial}{\partial x_i}, \tag{4.2}$$

then there exists a non-degenerate change of variables $\bar{x} = \bar{x}(t,x)$ which transforms the system into the form

$$\mathrm{d}\bar{x}_i = \bar{f}_i(t,\bar{x}_1,\ldots,\bar{x}_{n-1})\,\mathrm{d}t + \bar{g}_{i\alpha}(t,\bar{x}_1,\ldots,\bar{x}_{n-1})\,\mathrm{d}W_\alpha(t). \tag{4.3}$$

Thus, the system gets reduced to a system of $n-1$ SDEs for $\bar{x}_1,\ldots,\bar{x}_{n-1}$. The solution of the last equation can be given by quadratures

$$\bar{x}_n(t) = \bar{x}_n(t_0) + \int_{t_0}^{t} \bar{f}_i(s,\bar{x}_1(s),\ldots,\bar{x}_{n-1}(s))\,\mathrm{d}s + \int_{t_0}^{t} \bar{g}_{i\alpha}(s,\bar{x}_1(s),\ldots,\bar{x}_{n-1}(s))\,\mathrm{d}W_\alpha(s). \tag{4.4}$$

Theorem 4.2. *Suppose the system (2.1) admits an r-parameter solvable group of symmetries,*

$$X_k = \xi_i^k(t,x)\frac{\partial}{\partial x_i}, \qquad k = 1,\ldots,r, \tag{4.5}$$

acting regularly with r-dimensional orbits. Then the solution can be obtained by quadratures from the solution of a reduced system of order $n-r$. If the system (2.1) admits an n-parameter solvable group, its general solution can be found by quadratures.

A number of examples for integration of scalar SDEs with a one-dimensional Brownian motion were given in [13]. Let us consider a system of two SDEs in detail.

In the space of two variables (x_1, x_2) two-dimensional Lie algebras have five non-equivalent realizations by vector fields of the form

$$X = \xi_1(t, x_1, x_2)\frac{\partial}{\partial x_1} + \xi_2(t, x_1, x_2)\frac{\partial}{\partial x_2}, \tag{4.6}$$

which are given in table 1. According to corollary 2.7 three realizations, which are given by linearly connected operators, cannot be symmetries of the system (3.1), (3.2). Thus, we obtain the system

$$\begin{aligned} \mathrm{d}x_1 &= f_1(t)\,\mathrm{d}t + g_{1\alpha}(t)\,\mathrm{d}W_\alpha(t),\\ \mathrm{d}x_2 &= f_2(t)\,\mathrm{d}t + g_{2\alpha}(t)\,\mathrm{d}W_\alpha(t), \end{aligned} \tag{4.7}$$

which is invariant with respect to the operators

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = \frac{\partial}{\partial x_2} \tag{4.8}$$

and the system

$$\begin{aligned} \mathrm{d}x_1 &= f_1(t)x_2\,\mathrm{d}t + g_{1\alpha}(t)x_2\,\mathrm{d}W_\alpha(t),\\ \mathrm{d}x_2 &= f_2(t)x_2\,\mathrm{d}t + g_{2\alpha}(t)x_2\,\mathrm{d}W_\alpha(t), \end{aligned} \tag{4.9}$$

which admits the symmetries

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_1 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}. \tag{4.10}$$

It is easy to see that these systems are integrable by quadratures. The equations of the system (4.7) can be integrated independently. In the system (4.9) we can integrate the second equation as

$$x_2(t) = x_2(t_0)\exp\left(\int_{t_0}^t \left(f_2(s) - \frac{1}{2}g_{2\alpha}^2(s)\right)\mathrm{d}s + \int_{t_0}^t g_{2\alpha}(s)\,\mathrm{d}W_\alpha(s)\right)$$

and use this solution to integrate the first equation.

Example 4.2. Let us consider the system

$$\begin{aligned} \mathrm{d}x_1 &= (a_1 + b_1x_1 + c_1x_2)\,\mathrm{d}t + C_{11}\,\mathrm{d}W_1(t) + C_{12}\,\mathrm{d}W_2(t),\\ \mathrm{d}x_2 &= (a_2 + b_2x_1 + c_2x_2)\,\mathrm{d}t + C_{21}\,\mathrm{d}W_1(t) + C_{22}\,\mathrm{d}W_2(t), \end{aligned} \tag{4.11}$$

satisfying the full rank condition (3.2). We will look for symmetries of the form (4.6). Resolving determining equations (2.6)–(2.7), we obtain

$$\xi_1 = \xi_1(t), \qquad \xi_2 = \xi_2(t).$$

Substitution into the last set (2.5) gives

$$\xi_1'(t) = b_1\xi_1(t) + c_1\xi_2(t), \qquad \xi_2'(t) = b_2\xi_1(t) + c_2\xi_2(t). \tag{4.12}$$

The solution of the system is two dimensional. By theorem 4.2 the system (4.11) is integrable by quadratures. The system can be transformed to the form (4.7) because the symmetry group is Abelian. The solution of the system (4.12) can always be given as

$$\begin{pmatrix}\xi_1(t)\\ \xi_2(t)\end{pmatrix} = \alpha\begin{pmatrix}A_1(t)\\ B_1(t)\end{pmatrix} + \beta\begin{pmatrix}A_2(t)\\ B_2(t)\end{pmatrix}.$$

We will not provide detailed expressions for the functions $A_1(t)$, $B_1(t)$, $A_2(t)$ and $B_2(t)$ because it would require one to consider three deferent cases for roots of the characteristic polynomial. The admitted symmetries

$$X_1 = A_1(t)\frac{\partial}{\partial x_1} + B_1(t)\frac{\partial}{\partial x_2}, \qquad X_2 = A_2(t)\frac{\partial}{\partial x_1} + B_2(t)\frac{\partial}{\partial x_2}$$

can be transformed to the form (4.8) by the change of variables

$$\bar{x}_1 = \frac{B_2(t)x_1 - A_2(t)x_2}{\Delta}, \qquad \bar{x}_2 = \frac{-B_1(t)x_1 + A_1(t)x_2}{\Delta},$$

where

$$\Delta = A_1(t)B_2(t) - A_2(t)B_1(t).$$

The transformation brings the system (4.11) to the form

$$\begin{aligned} \mathrm{d}\bar{x}_1 &= f_1(t)\,\mathrm{d}t + g_{11}(t)\,\mathrm{d}W_1(t) + g_{12}(t)\,\mathrm{d}W_2(t), \\ \mathrm{d}\bar{x}_2 &= f_2(t)\,\mathrm{d}t + g_{21}(t)\,\mathrm{d}W_1(t) + g_{22}(t)\,\mathrm{d}W_2(t) \end{aligned}$$

with

$$\begin{aligned} f_1(t) &= \frac{B_2(t)a_1 - A_2(t)a_2}{\Delta}, & f_2(t) &= \frac{-B_1(t)a_1 + A_1(t)a_2}{\Delta}, \\ g_{11}(t) &= \frac{B_2(t)C_{11} - A_2(t)C_{21}}{\Delta}, & g_{21}(t) &= \frac{-B_1(t)C_{11} + A_1(t)C_{21}}{\Delta}, \\ g_{12}(t) &= \frac{B_2(t)C_{12} - A_2(t)C_{22}}{\Delta}, & g_{22}(t) &= \frac{-B_1(t)C_{12} + A_1(t)C_{22}}{\Delta}. \end{aligned}$$

The subsequent integration is straightforward.

Remark 4.3. A wider family of SDEs can be integrated if we consider realizations of SDEs by decoupled systems [26, 27]. Since this integration method goes beyond transformations (2.10), it is not considered in this paper. It is worth mentioning that besides quadratures there are other means to present closed-form solutions of SDEs [28].

Although there are more powerful methods for integration of SDEs based on decoupling, symmetry methods suggest changes of variables which lead to the simplification of systems of SDEs. In many cases it can be sufficient for integration.

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